Editor's Note: This first appeared on the web in 2003 and is included in the Making Sense archive because, well, just because. –j.j.

What Is The Deal With Fibonacci Numbers, Anyway?

After reading the book, "The DaVinci Code" by Dan Brown (it's FICTION, people!), I've been doing a little thinking on the subject of Fibonacci numbers. I've also received several inquiries on the subject from people who would like help making some sense out of it.

Fibonacci Numbers are a series of numbers made by adding the two previous entries to obtain the next. They are named after a famous Italian named Fibonacci who became famous for creating a sequence of numbers formed by adding the two previous numbers to get the next number in the sequence. Back in those days, evidently, it didn't take much to get noticed.

For example, if you start with 0 and 1, you get the well-known (famous even, among your nerdier demographics) Fibonacci Sequence:

0 1 1 2 3 5 8 13 21 34 55 89 . . .

"So what?" you say. Well, the curious thing about this little sequence is that very soon, the ratio of any two consecutive terms begins to converge to the value 1.61803398874989... . Don't believe me? Try it at home:

8/5 = 1.6
13/8 = 1.625
21/13 = 1.615
34/21 = 1.619
55/34 = 1.618
89/55 = 1.618

"Again," you say, "So What?" Aha. This is where the Conspiracy Theory starts. This same ratio happens to occur all over in Nature. You've heard of Nature before, haven't you? It's that big thing, outdoors.

The human body, as another famous Italian Leonardo Da Vinci is famous for pointing out, exhibits this precise ratio in numerous places. It also shows up in snail shells, flower petals, seed heads, crystals or virtually anywhere Nature makes a recursive geometric pattern. So, maybe there's something special about that sequence of numbers. The number 1.61803398874989... is sometimes called the Divine Ratio. Supposedly, when He created everything, God preferred this particular number for some reason.

But, what is so special about the Fibonacci sequence? Are there not other sequences of numbers you could get by a similar process? Why don't THEY produce important ratios favored by Divine Beings? Hmmm? I asked myself this question recently, and set out trying to produce alternatives to the Fibonacci sequence.

For the ratio to have any chance of being different, the sequence must be more than just a scaled-up version of the original, that is, every number in the series multiplied by 2, 3, 4, etc. The sequence starting with 0,2 for example, is exactly like the original 0,1 sequence, except every term is doubled. You get: 0 2 2 4 6 10 16 26 . . . which results in the same Divine Ratio. Likewise with 0,3 (every term tripled), and so on. Starting with 1,2, you again get the original sequence, because the original contains 1 and 2 as neighbors, and the sequence picks right up from that point. That eliminates any pair of numbers that appears in the original sequence, as well as in their doubles, triples, etc. as potentially creating truly unique sequences. For example, 2,3 or 3,5 or 4,6 would all produce essentially the same sequence or multiples thereof, and therefore the same Divine Ratio.

But what about, say, 1 and 3? Let's see: 1 3 4 7 11 18 29 47 . . . Yes, this appears to be a truly different sequence, not merely a scaled or shifted version of the original Fibonacci sequence. But what ratio does it produce? 47/29 = 1.62. After a paltry 12 or so terms, you get 1.6180... . Precisely equal to the Divine Ratio. After 20 or so steps, you get the ratio to an Absurd degree of accuracy. After a mere 40 or so steps, the accuracy surpasses Insane level and heads straight into Ludicrous territory. It turns out, that any sequence you can make by adding the two preceding terms to get the next term, no matter where or how the sequence started, eventually creates the Divine Ratio. So what is going on here? What is so Divine about this number 1.61803398874989...?

I will attempt to answer that. But first, a proof. I stated that ANY sequence made by adding two consecutive numbers to obtain the next number has the result that the ratio of consecutive numbers approaches a constant. And not just any old constant, either. It creates none other than the Divine Ratio. Here is proof of this astonishing fact.

Suppose I have some arbitrary list of numbers in which every number is the sum of the previous two numbers in the list. Also, suppose that the ratio of any two terms in the series approaches some fixed, but unknown (for the present) value, a ratio I'll call 'R'. Somewhere in the middle of this series is some number I'll call A. The next number is A plus the previous number. But it is also A times the ratio R: AR. The next term after that is A + AR, or after re-grouping, A(R+1). The ratio of A(R+1) to AR is (R+1)/R. But we have already said that the ratio of any two consecutive terms is a constant, so (R+1)/R must be equal to R. This leads to the equation (R+1)/R = R, or in standard polynomial form,

$$\mathbf{R^2 - R - 1 = 0.}$$

This equation must be true if our initial assumptions are correct. Solving for a positive value of R:

R = (1+sqrt(1+4))/2 = 0.5 + sqrt(5/4) = 1.61803398874989... Look familiar?

Therefore any sequence with the properties I stated at the beginning of the proof MUST produce the Divine Ratio, as a mathematical necessity. But why, you rightly ask, does every sequence formed by adding previous consecutive terms produce any fixed ratio at all, let alone the Divine Ratio? What happens if we do not assume that the ratio of consecutive terms is a constant?

Suppose we have some sequence in which A and AR_1 are two consecutive terms. 'R_1' is not necessarily constant - it's just a result. The next term will be $A+AR_1$. The ratio of that term to the previous is $(R_1+1)/R_1$, and this ratio is equal to some, possibly different, ratio R_2. This produces the equation

$$(R_1 -1)/R_1 = R_2.$$

We have already seen that if R_1 and R_2 are equal, then they are both equal to the Divine Ratio. But if R_1 is larger than the Divine Ratio, then the equation above shows that R_2 will be smaller than the Divine Ratio, but NOT BY AS MUCH! Similarly, if R_1 is smaller than the Divine Ratio, then R_2 must be larger, but again, by a smaller difference. As the sequence progresses, the ratios alternate between values higher and lower than the Divine Ratio while constantly getting closer and closer to it until they are practically equal to the Divine Ratio. That means that the ratio of any two consecutive terms in this kind of sequence might start out being anything, but must soon converge on the value 1.61803398874989... with ridiculous speed.

The Divine Ratio is not really such a conspiracy after all: it's just the solution to the rather dull and ordinary equation $(R+1)/R = R$.

So, why does it occur so often in that big, non-virtual thing outdoors, aka Nature? It isn't too hard to imagine how growth processes in Nature could be modeled by this type of recursive equation: A snail can only be as big as its shell allows; and its shell can only get larger by an incremental amount proportional to the size of the snail. The size of the shell equals the previous shell plus the incremental growth. Voila: a Fibonacci series and the Divine Ratio. The fact that this ratio appears in so many places says more about the cumulative and recursive nature of growth than about the Divine nature of an old list of numbers, or a conspiracy among old Italians.